HOMOTOPICALLY NON-TRIVIAL MAPS WITH SMALL K-DILATION

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Abstract. We construct homotopically non-trivial maps from S^m to S^n with arbitrarily small 3-dilation for certain pairs (m, n). The simplest example is the case $m = 4, n = 3$, and there are other pairs with arbitrarily large values of both m and n. We show that a homotopy class in $\pi_7(S^4)$ can be represented by maps with arbitrarily small 4-dilation if and only if the class is torsion.

The k-dilation of a map measures how much the map stretches k-dimensional volumes. If f is a C^1 map between Riemannian manifolds, we say that the k-dilation of f is at most D if f maps each k-dimensional submanifold of the domain with volume V to an image with volume at most DV. We get the same k-dilation whether we consider all submanifolds or whether we consider only small disks, and so the k-dilation can also be defined in terms of the first derivative df. Recall that $\Lambda^k df$, the k-fold exterior product of the derivative df, maps $\Lambda^k TM$ to $\Lambda^k TN$. If f is C^1, the k-dilation of f is equal to the supremal value of the norm $|\Lambda^k df|$.

In this paper, we examine to what extent a bound on the k-dilation of a map controls the homotopy type of the map. A beautiful result of this kind was recently obtained by Tsui and Wang in [7].

Theorem. *(Tsui and Wang) Let f be a C^1 map from S^m to S^n, where $m \geq 2$. If the 2-dilation of f is less than 1, then f is nullhomotopic.*

The main result of this paper shows that the situation is very different for 3-dilation.

Theorem 1. *For each n, there are infinitely many m so that the following holds: there are homotopically non-trivial maps from S^m to S^n with arbitrarily small 3-dilation.*

This result partly answers a question raised by Gromov in [5] (page 231). Gromov asked for which values of k, q, m, and n is a map $f : S^m \to S^n$ with a sufficiently small norm $|\Lambda^k df|_{L^q}$ necessarily null-homotopic.

We make the following definition. A homotopy class a in $\pi_m(S^n)$ lies in $V_k\pi_m(S^n)$ if there are maps in the homotopy class a with arbitrarily small k-dilation. We will prove that $V_k\pi_m(S^n)$ is a subgroup of $\pi_m(S^n)$ and that $V_k\pi_m(S^n) \subset V_{k+1}\pi_m(S^n)$. Therefore, $V_k\pi_m(S^n)$ defines a filtration of $\pi_m(S^n)$. More generally, we will define a filtration $V_k\pi_m(X)$ for any space X.

Our methods give some partial information about the filtration $V_k\pi_m(S^n)$. The information is most interesting for the filtration $V_k\pi_7(S^4)$. Recall that $\pi_7(S^4)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_{12}$.

Theorem 2. *The group $V_4\pi_7(S^4)$ is the torsion subgroup of $\pi_7(S^4)$. It is a proper, non-zero subgroup.*

The paper is organized as follows. In the first section, we prove the first theorem. In the second section, we summarize some lower bounds for k-dilation that appear in the literature. In the third section, we define $V_k \pi_m(X)$, and prove its basic algebraic properties. In the fourth section, we look at some examples, including $V_k \pi_7(S^4)$.

This paper is based on a section of my thesis. I would like to thank my thesis advisor, Tom Mrowka, for his help and support.

1. Homotopically non-trivial maps with small k-dilation

This section contains the main result of the paper. It gives a construction for maps between spheres with very small k-dilation which are homotopic to high order suspensions.

Proposition 1. *Fix a homotopy class a in $\pi_m(S^n)$ and then consider its p-fold suspension $\Sigma^p a$ in $\pi_{m+p}(S^{n+p})$, and let k be an integer greater than $n + (n/m)p$. Then there are maps from S^{m+p} to S^{n+p} in the homotopy class $\Sigma^p a$ with arbitrarily small k-dilation.*

Proof. Let f_1 be a map in the homotopy class a from $[0,1]^m$ to the unit n-sphere, taking the boundary of the domain to the basepoint of S^n. Let f_2 be a degree 1 map from $[0,1]^p$ to the unit p-sphere, taking the boundary of the domain to the basepoint of S^p. We can assume both maps are C^1, and we pick a number L which is bigger than the Lipshitz constant of either map.

Inside of the unit (m+p)-sphere, we can quasi-isometrically embed a rectangle R with dimensions $[0,\epsilon]^m \times [0,\epsilon^{-m/p}]^p$. (The quasi-isometric constant does not depend on ϵ.) Now we construct a map F from R to $S^n \times S^p$. The map F is a direct product of a map F_1 from $[0,\epsilon]^m$ to S^n and a map F_2 from $[0,\epsilon^{-m/p}]^p$ to S^p. The map F_1 is just a dilation from $[0,\epsilon]^m$ to the unit cube, composed with the map f_1. The map F_2 is just a dilation from $[0,\epsilon^{-m/p}]^p$ to $[0,1]^p$, composed with the map f_2.

When k is bigger than n, the k-dilation of F is less than $(L\epsilon^{-1})^n (L\epsilon^{m/p})^{k-n}$. Expanding this expression gives $L^k \epsilon^{-n+(m/p)k-(m/p)n}$. The important part of the expression is the power of ϵ, which is equal to $(m/p)(k - n - (n/m)p)$. We have assumed that k is greater than $n + (n/m)p$, and so the exponent of ϵ is positive. For ϵ sufficiently small, the k-dilation of F is arbitrarily small.

The map F takes the boundary of R to $S^n \vee S^p$. We compose F with a smash map, which is a degree 1 map from $S^n \times S^p$ to S^{n+p}, taking $S^n \vee S^p$ to the base point. The result is a map from R to S^{n+p} which takes the boundary of R to the basepoint. We can easily extend this map to all of S^{m+p} by mapping the complement of R to the basepoint of S^{n+p}. The resulting map is homotopic to $\Sigma^p(a)$, and it has arbitrarily small k-dilation. □

We can apply our proposition to the non-trivial homotopy class in $\pi_4(S^3)$, which is represented by the suspension of the Hopf fibration. In this case, we are considering a 1-fold suspension of a map from S^3 to S^2. Therefore $p = 1$, $m = 3$, and $n = 2$. Since $3 > 2 + (2/3)1$, this homotopy class can be realized by maps from S^4 to S^3 with arbitrarily small 3-dilation.

We now turn to the proof of our main theorem, which gives infinitely many examples of non-trivial homotopy classes that can be realized with arbitrarily small 3-dilation. The proof of this theorem uses some deep results in algebraic topology

to guarantee that certain high-order suspensions are homotopically non-trivial. I would like to thank Haynes Miller for helping me to find the relevant results in the topology literature.

Theorem 1. *For every $N \geq 2$, there are infinitely many M so that there are homotopically non-trivial maps from S^M to S^N with arbitrarily small 3-dilation.*

Proof. When $i = 8j + 1$, the homotopy group $\pi_i(SO)$ is equal to $\mathbb{Z}_2$. The stable J-homomorphism maps $\pi_i(SO)$ to the i^{th} stable stem of the homotopy groups of spheres. The image of the J-homomorphism was studied by Adams in [1]. When $i = 8j + 1$, he proved that the stable J-homomorphism is injective. Its image is a copy of $\mathbb{Z}_2$. This image contains a non-trivial element in $\pi_{i+n}(S^n)$, for large n. It turns out that this non-trivial element is the suspension of a class in $\pi_{i+2}(S^2)$. This statement is made clearly in the introduction to the paper [3], and the proof appears in the older paper [2]. For each $i = 8j + 1$, let a_i be a homotopy class in $\pi_{i+2}(S^2)$ whose suspension is the non-trivial element in the image $J(\pi_i(SO))$. In particular, the p-fold suspension $\Sigma^p a_i$ is non-trivial for every p and every i.

Now let N be any integer at least 2. For each $i = 8j + 1$ greater than $2N - 6$, consider the class $\Sigma^{N-2} a_i$ in $\pi_{i+N}(S^N)$. We let $M = N + i$. Each of these homotopy classes is non-trivial. In the language of Proposition 1, we have $p = N - 2$, $m = i + 2$, and $n = 2$. The condition that i is greater than $2N - 6$ exactly guarantees that $3 > n + (n/m)p$. Therefore, each of these homotopy classes can be realized with arbitrarily small 3-dilation. □

This proof constructs homotopically non-trivial maps from S^M to S^N with arbitrarily small 3-dilation for many pairs (M, N). For example, when $N = 3$, we can take $M = 4, 12, 20, 28, 36$, and so on; and when $N = 4$, we can take $M = 13, 21, 29, 37$, and so on.

2. Lower bounds for k-dilation

In this section, we survey several lower bounds for the k-dilation of mappings. We begin by recalling the theorem of Tsui and Wang mentioned in the introduction.

Theorem. *(Tsui and Wang, [7]) Let f be a C^1 map from S^m to S^n, where $m \geq 2$. If the 2-dilation of f is less than 1, then f is nullhomotopic.*

The proof by Tsui and Wang uses the mean curvature flow to deform the graph of the map f as a submanifold of $S^m \times S^n$. They prove that the mean curvature flow converges to the graph of a constant function and that at each time t the flowed submanifold is the graph of a map f_t. Therefore, f_t provides a homotopy from t to a constant map.

Earlier, Gromov had proved a slightly weaker theorem in the same spirit. He proved that for each m and n, there exists a number $\epsilon(m, n) > 0$, so that any C^1 map from S^m to S^n with 2-dilation less than $\epsilon(m, n)$ is null-homotopic. Gromov's proof is completely different from the proof of Tsui and Wang. He approaches the map from S^m to S^n as a family of maps from S^2 to S^n. If the 2-dilation is less than ϵ, then each image of S^2 has area less than $4\pi\epsilon$. Gromov uses the Riemann mapping theorem to change coordinates on the domain so that each map in the family has Dirichlet energy less than $C\epsilon$. Then he uses the borderline Sobolev inequality, which bounds the BMO norm of each map in the family by $C\epsilon$. Using the bound on the BMO norm, he constructs a homotopy from the initial family of

maps to a family of constant maps. This argument is sketched on page 179 of the long essay [5].

Gromov's estimates are less sharp than those of Tsui and Wang, but he is able to push them through with much weaker hypotheses. For example, Gromov proves that if f maps S^m to S^n with $|\Lambda^2 df|_{L^{m-1+\delta}} < \epsilon(m,n,\delta)$ then f is null-homotopic. Gromov's method also applies to more manifolds. He proves that for any two compact simply connected Riemannian manifolds M and N there is a constant $\epsilon(M,N)$ so that any map $f : M \to N$ with 2-dilation less than $\epsilon(M,N)$ is null-homotopic. (Both these results are special cases of the Corollary on page 230 of [5].)

Lower bounds for the k-dilation with k greater than 2 are much rarer. We begin with a very elementary example. If a C^1 map f from S^n to S^n is homotopically non-trivial, then it has n-dilation at least 1. This result follows because a map with n-dilation equal to $1-\epsilon$ has an image with volume at most $(1-\epsilon)Vol(S^n)$. Such a map is not surjective, and so it is null-homotopic.

Gromov also proved lower bounds for the k-dilation of maps with non-zero Hopf invariant. The following argument appears in [4] on pages 358-359.

Theorem. *(Gromov) Let f be a map from S^{4n-1} to S^{2n} with $2n$-dilation D. Then the norm of the Hopf invariant of f is bounded by $C(n)D^2$. Since the Hopf invariant is an integer, any map with non-zero Hopf invariant has $2n$-dilation at least $C(n)^{-1/2}$.*

Proof. Let ω be a 2n-form on S^{2n} with $\int \omega = 1$. The pullback $f^*(\omega)$ is a closed 2n-form on S^{4n-1}. Since $H^{2n}(S^{4n-1}) = 0$, this form is exact. We let $Pf^*(\omega)$ denote any primitive of $f^*(\omega)$. Then the Hopf invariant of f is equal to $\int_{S^{4n-1}} Pf^*(\omega) \wedge f^*(\omega)$.

During this proof, we let C denote a constant depending on n that may change from line to line. We can assume that $|\omega| < C$ at every point of S^{2n}. The norm of $f^*(\omega)$ is bounded by CD pointwise. Therefore, the L^2 norm of $f^*(\omega)$ is bounded by CD. Using Hodge theory, we can choose $Pf^*(\omega)$ to be perpendicular to all of the exact (2n-1)-forms. For this choice, the L^2 norm of $Pf^*(\omega)$ is bounded by $\lambda^{-1/2}CD$, where λ is the smallest eigenvalue of the Laplacian on exact (2n)-forms. The eigenvalue λ is greater than zero and depends only on n. Finally, the norm of the Hopf invariant is bounded by $|f^*(\omega)|_{L^2}|Pf^*(\omega)|_{L^2}$, which is bounded by $C(n)D^2$. □

The method above can be generalized to many non-torsion homotopy classes. For more information, see Gromov's discussion in [5], pages 220-223.

The methods outlined here leave many open questions. If a is a torsion homotopy class in some homotopy group $\pi_m(S^n)$, I don't know any way to lower bound the 3-dilation of maps in the homotopy class a. For example, the 100-fold suspension of the Hopf map is a non-trivial element in $\pi_{103}(S^{102})$. According to Proposition 1, it can be realized by maps with arbitrarily small 69-dilation. It would be interesting to know whether it can be realized by maps with arbitrarily small 3-dilation.

3. A filtration on homotopy groups

In this section, we define a filtration $V_k\pi_m(X)$ which measures which homotopy classes in $\pi_m(X)$ can be realized with arbitrarily small k-dilation. We derive some

easy formal properties of this filtration. In the next section, we will calculate it in some examples and show that it can be non-trivial.

As a first step, we define $V_k \pi_m(W)$ when W is a finite simplicial complex. We put a piecewise Riemannian metric on W by equipping each simplex with the metric of an equilateral simplex in Euclidean space with edges of length 1. We then define $V_k \pi_m(W)$ to be the set of homotopy classes in $\pi_m(W)$ that can be realized by maps $S^m \rightarrow W$ with arbitrarily small k-dilation.

The set $V_k \pi_m(W)$ is in fact a subgroup of $\pi_m(W)$. If a lies in $V_k \pi_m(W)$, then let f_i be a sequence of maps from S^m to W in the homotopy class a with k-dilation tending to zero. Let I be a reflection, mapping S^m to itself with degree -1, and taking the basepoint of S^m to itself. Then the maps $f_i \circ I$ have k-dilations tending to zero and lie in the homotopy class $-a$. Therefore $-a$ lies in $V_k \pi_m(W)$. Next, suppose that a and b lie in $V_k \pi_m(W)$. Again, let f_i be a sequence of (pointed) maps in the class a with k-dilation tending to zero, and let g_i be a sequence of (pointed) maps in the homotopy class b with k-dilation tending to zero. Let I be a map from S^m to $S^m \vee S^m$ with degree (1,1). Let h_i be the map from $S^m \vee S^m$ to W whose restriction to the first copy of S^m is equal to f_i and whose restriction to the second copy of S^m is equal to g_i. Then the sequence $h_i \circ I$ has k-dilation tending to zero. Each map in the sequence lies in the homotopy class $a + b$. So $a + b$ lies in $V_k \pi_m(W)$.

The sets $V_k \pi_m(W)$ are nested, with $V_k \pi_m(W) \subset V_{k+1} \pi_m(W)$. To see this, we express the k-dilation of a map in terms of the singular values of its derivative. Let f be a piecewise C^1 map from S^m to W. Suppose that the singular values of df at a point are equal to $0 \leq s_1 \leq ... \leq s_m$. Then the norm $|\Lambda^k df|$ at that point is equal to $s_{m-k+1} ... s_m$. Therefore, $|\Lambda^{k+1} df| \leq |\Lambda^k df|^{\frac{k+1}{k}}$. If a map f has k-dilation D, then the (k+1)-dilation of f is at most $D^{\frac{k+1}{k}}$. In particular, if f_i is a sequence of maps with k-dilation tending to zero, then the (k+1)-dilation of f_i also tends to zero. Therefore, $V_k \pi_m(W) \subset V_{k+1} \pi_m(W)$. Any map from S^m has $(m+1)$-dilation zero, and so $V_{m+1} \pi_m(W)$ is always the whole homotopy group $\pi_m(W)$.

The last paragraph shows that the sequence of groups $V_k \pi_m(W)$ is a filtration of $\pi_m(W)$.

$0 = V_1 \pi_m(W) \subset V_2 \pi_m(W) \subset ... \subset V_m \pi_m(W) \subset V_{m+1} \pi_m(W) = \pi_m(W)$.

The filtration V_k behaves naturally under mappings in the following sense. If $F : W \rightarrow V$ is a continuous pointed mapping between finite simplicial complexes, then $F_* : \pi_m(W) \rightarrow \pi_m(V)$ takes $V_k \pi_m(W)$ into $V_k \pi_m(V)$. To prove this, first approximate F by a PL map with some finite Lipshitz constant L. Let a be a class in $V_k \pi_m(W)$, realized by mappings $f_i : S^m \rightarrow W$ with k-dilation tending to zero. The map $F \circ f_i$ from S^m to V has k-dilation less than L^k times the k-dilation of f_i, so the the sequence $F \circ f_i$ has k-dilation tending to zero. Each map $F \circ f_i$ lies in the homotopy class $F_*(a)$. Therefore, $F_*(a)$ lies in $V_k \pi_m(V)$.

Now we define $V_k \pi_m(X)$ for an arbitrary space X. A homotopy class a in $\pi_m(X)$ belongs to $V_k \pi_m(X)$ if there is a finite simplicial complex W and a map $F : W \rightarrow X$ so that a lies in the image $F_*(V_k \pi_m(W))$. In case X is a finite simplicial complex, this definition agrees with our first definition because of the mapping property of V_k proved in the last paragraph. As above, $V_k \pi_m(X)$ defines a filtration of $\pi_m(X)$. Also, if $f : X \rightarrow Y$ is any continuous map of spaces, then f_* maps $V_k \pi_m(X)$ into $V_k \pi_m(Y)$. From this mapping property, it follows that the filtration $V_k \pi_m(X)$ is a homotopy invariant.

We can rephrase the main theorems of this paper in the language of the filtration $V_k\pi_m(S^n)$. The theorem of Tsui and Wang implies that $V_2\pi_m(S^n) = 0$ for $m \geq 2$. Theorem 1 says that for each $n \geq 2$, the group $V_3\pi_m(S^n)$ is non-zero for infinitely many m. Theorem 2 says that $V_4\pi_7(S^4)$ is exactly the torsion subgroup of $\pi_7(S^4)$, which is isomorphic to $\mathbb{Z}_{12}$.

4. Examples of the V_k Filtration

In this section, we use the tools from sections 1 and 2 to compute the filtration $V_k\pi_m(S^n)$ for a few small values of m and n. For other values of m and n, we obtain some partial information about the filtration. The most interesting example is a computation of $V_4\pi_7(S^4)$. We use the lists of homotopy groups of spheres and the suspension maps between them given in [6] on pages 39-42. According to the the theorem of Tsui and Wang, $V_2\pi_m(S^n)$ is zero in all the examples below, so we will only discuss V_k for $k \geq 3$.

The filtration of $\pi_m(S^2)$

Any element of $\pi_m(S^2)$ lies in $V_3\pi_m(S^2)$, because any smooth map from S^m to S^2 has 3-dilation equal to zero.

The filtration of $\pi_4(S^3)$

The group $\pi_4(S^3)$ is isomorphic to $\mathbb{Z}_2$, and the non-trivial element is given by the suspension of the Hopf fibration from S^3 to S^2. Applying Proposition 1, this non-trivial element lies in $V_3\pi_4(S^3)$.

The filtration of $\pi_5(S^3)$

The group $\pi_5(S^3)$ is isomorphic to $\mathbb{Z}_2$, and the non-trivial element is the suspension of a class in $\pi_4(S^2)$. Applying Proposition 1, this non-trivial element lies in $V_3\pi_5(S^3)$.

The filtration of $\pi_6(S^3)$

The group $\pi_6(S^3)$ is isomorphic to $\mathbb{Z}_{12}$. One non-trival element is the suspension of a class in $\pi_5(S^2)$. Applying Proposition 1, this non-trivial element lies in $V_3\pi_6(S^3)$. I don't know whether the other non-trivial elements lie in $V_3\pi_6(S^3)$.

The filtration of $\pi_5(S^4)$

The group $\pi_5(S^4)$ is isomorphic to $\mathbb{Z}_2$, and the non-trivial element is given by the double suspension of the Hopf fibration from S^3 to S^2. Applying Proposition 1, this non-trivial element lies in $V_4\pi_5(S^4)$. I don't know whether it lies in $V_3\pi_5(S^4)$.

The filtration of $\pi_6(S^4)$

The group $\pi_6(S^4)$ is isomorphic to $\mathbb{Z}_2$, and the non-trivial element is the double suspension of a class in $\pi_4(S^2)$. Applying Proposition 1, this non-trivial element lies in $V_4\pi_6(S^4)$. I don't know whether it lies in $V_3\pi_6(S^4)$.

The filtration of $\pi_7(S^4)$

The group $\pi_7(S^4)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_{12}$. The Hopf invariant gives a map $H : \pi_7(S^4) \to \mathbb{Z}$. In section 2, following Gromov, we proved that $V_4\pi_7(S^4)$ lies in the kernel of H. The kernel of H is isomorphic to $\mathbb{Z}_{12}$. The suspension map is an isomophism from $\pi_6(S^3)$ onto the kernel of H, as follows from the long exact sequence of the Hopf fibration $S^3 \to S^7 \to S^4$. (This fact is stated in [6] on page 2.) Therefore, every element in the kernel of H is a suspension. Applying Proposition 1, we see that the kernel of H lies in $V_4\pi_7(S^4)$. Therefore, $V_4\pi_7(S^4)$ is exactly the kernel of H. This proves the second theorem in the introduction.

In addition, one non-trivial element in $\pi_7(S^4)$ is a double suspension of an element in $\pi_5(S^2)$. Applying Proposition 1 to this element, we see that it lies in $V_3\pi_7(S^4)$. I don't know whether the other torsion elements of $\pi_7(S^4)$ lie in $V_3\pi_7(S^4)$.

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